Coherent Oscillations of a Ring of Relativistic Particles

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Abstract

The effect of ring curvature on the coherent perturbations of a ring of relativistic particles is studied within the framework of the linearized Vlasov equation. Finite curvature is shown to have a minor effect on the dynamics of the "negative mass" mode; the "transverse" mode in radial direction, however, is found to be coupled with a simultaneous longitudinal density modulation which modifies the dispersion relation. In the limit of small mode frequency ($\omega/\Omega \ll 1$) Landau damping of the resistive wall or electron ion instability is shown to require a threshold dispersion which sensitively depends on the effect of curvature.

1. Introduction

Longitudinal and transverse coherent oscillations of intense particle beams have been investigated by many authors with respect to the negative mass instability [1,2], the longitudinal and transverse resistive instabilities [3,4] and the electron-ion instability [5], which may be important limiting factors for accelerators with pronounced collective behaviour.

In the existing literature these instabilities are usually treated within the limit of negligible beam curvature. This straight-beam limit is an approximation sufficient to describe many real situations. Rings of highly relativistic electrons, however, like those used in the electron ring accelerator (ERA) devices are significantly influenced by curvature effects, unless there is strong cancellation of the electron space charge by an ion background. In the Garching ERA device [6] axial focussing of the ring in the accelerating structure was found to be possible only if the axially defocussing effect of curvature is compensated by images on an electric image cylinder ("squirrel cage") close to the ring. Not only the equilibrium but also the stability properties of such electron rings may critically depend on curvature effects. In particular one must expect coupling of longitudinal and radial coherent oscillations owing to the centrifugal force term in the single particle equations of motion. This paper treats the analysis of coherent modes within the framework of the relativistic Vlasov equation in cylindrical geometry. In contrast with previous works curvature is correctly taken into account in the Vlasov analysis. The curvature-modified dispersion relations of the most important modes are derived. In a special case (low frequency radial mode) it is shown that Landau damping by a finite energy spread may be suppressed by curvature effects, which renders the mode (linearly) unstable with respect to the resistive wall and electron-ion instability.

2. Basic Equations and Equilibrium

The distribution in phase space of one species of relativistic particles with charge e interacting through their collective electric and magnetic field is described by the relativistic Vlasov equation in cylindrical geometry [7,8]*:

(1) $$\frac{\partial f}{\partial t} + \dot{r}\frac{\partial f}{\partial r} + \dot{u}_r\frac{\partial f}{\partial u_r} + \dot{\theta}\frac{\partial f}{\partial \theta} + \dot{u}_\theta\frac{\partial f}{\partial u_\theta} + \dot{z}\frac{\partial f}{\partial z} + \dot{u}_z\frac{\partial f}{\partial u_z} = 0$$

(2)
$$u_r = \gamma\dot{r}$$
$$u_\theta = \gamma v_\theta = \gamma r\dot{\theta}$$
$$u_z = \gamma\dot{z}$$
$$\gamma^2 = 1 + u_r^2 + u_\theta^2 + u_z^2$$
$$\dot{u}_r = \frac{u_\theta^2}{r\gamma} + \frac{e}{m}\left(E_r + \frac{u_\theta}{\gamma}B_z - \frac{u_z}{\gamma}B_\theta\right); \quad \dot{u}_\theta = -\frac{u_\theta u_r}{r\gamma} + \frac{e}{m}\left(E_\theta - \frac{u_r}{\gamma}B_z + \frac{u_z}{\gamma}B_r\right)$$
$$\dot{u}_z = \frac{e}{m}\left(E_z - \frac{u_\theta}{\gamma}B_r + \frac{u_r}{\gamma}B_\theta\right)$$

with electric and magnetic fields satisfying the Maxwell equations with appropriate boundary conditions

(3)
$$\mathrm{div}\ \underline{E} = 4\pi e n; \quad \mathrm{curl}\ \underline{B} = 4\pi\underline{j} + \frac{\partial \underline{E}}{\partial t}$$
$$\mathrm{curl}\ \underline{E} = -\frac{\partial \underline{B}}{\partial t}; \quad \mathrm{div}\ \underline{B} = 0$$

and particle density and current density computed selfconsistently with the Vlasov equation

(4)
$$n(r,\theta,z,t) = \int f\, du_r\, du_\theta\, du_z$$
$$\underline{j}(r,\theta,z,t) = e\int \frac{\underline{u}}{\gamma} f\, du_r\, du_\theta\, du_z$$

A rotationally symmetric equilibrium distribution can be described in terms of two constants of the motion, the total energy and the canonical angular momentum [9])

* with units such that the speed of light becomes unity

(5) $H^0 = m\,\gamma + e\,\Phi^0$

(6) $P_\theta = m\,r\,u_\theta + e\,r\,A^0_\theta$

(7) $f^0 = f^0(H^0, P_\theta)$

Next we replace u_θ by P_θ according to (6). The energy $H^0(r,z,u_r,P_\theta,u_z)$ is in general a complicated function of its variables. With regard to the linearization of the problem it is desirable to expand H^0 about the minimum value of energy for a given P_θ. The variational problem

(8) $\delta H = 0$

yields a solution

(9) $u_r \equiv u_z \equiv 0;\ r \equiv R(P_\theta);\ z \equiv 0$

if there is symmetry in z and no applied E_z-field.

This corresponds to purely circular motion at radius $R(P_\theta)$ with zero order values $u_{o\theta}(P_\theta)$, $\gamma_o(P_\theta)$, $H^0_o(P_\theta)$ defined by

(10)
$$\begin{aligned} P_\theta &= mRu_{o\theta} + eRA^0_\theta(R,0) \\ \frac{m\,u_{o\theta}^2}{\gamma_o R} &+ \frac{e}{\gamma_o}u_{o\theta}B^0_z(R,0) + eE^0_r(R,0) = 0 \\ \gamma_o^2 &= 1 + u_{o\theta}^2 \\ H_o^2 &= m\gamma_o + e\Phi^0(R,0) \end{aligned}$$

We are now able to expand H^0 about H^0_o if we define the relative coordinate

(11) $x \equiv r - R(P_\theta)$

and obtain

$$(12) \quad H^0 = H_o^0(P_\theta) + H^0(x,u_r,z,u_z,P_\theta)$$

$H_\perp^0$ is the transverse energy which can be written in leading order as

$$(13) \quad H_\perp^0 \equiv \alpha_1 x^2 + \alpha_2 u_r^2 + \alpha_3 z^2 + \alpha_4 u_z^2$$

with $\alpha_i = \alpha_i(P_\theta)$ $(i = 1...4)$.

(13) holds for small betatron oscillation amplitudes which we shall assume in the following. We observe that in this order $H_\perp^0$ is also a constant of the motion. In addition it can be written as sum of the radial and axial oscillation energies which are constants as well:

$$(14) \quad \begin{aligned} H_{\perp r}^{\,0} &\equiv \alpha_1 x^2 + \alpha_2 u_r^2 \\ H_{\perp z}^{\,0} &\equiv \alpha_3 z^2 + \alpha_4 u_z^2 \end{aligned}$$

We can now specify f^0 as a function

$$(15) \quad f^0 = f^0(P_\theta, H_{\perp r}^{\,0}, H_{\perp z}^{\,0})$$

with the following restrictions

(16)

(a) f^0 non-negative

(b) f^0 is zero outside a finite range of P_θ values (such that $0 < R_{min} \leqq R(P_\theta) \leqq R_{max}$)

(c) f^0 is non-vanishing only within ranges of values of x and z sufficiently small to justify omission of higher than second order terms in (13).

We remark that the approximate equilibrium (15) is more realistic than the axact solutions (7), which distribute particles uniformly on energy surfaces in six-dimensional phase space. There is sufficient evidence that besides P_θ, H^0 there exists a further constant of the motion, although such a constant may not be expressible as an analytic function of the phase space coordinates. Replacing H^0 by $H_{\perp r}^0$, $H_{\perp z}^0$ we give at least an approximate account to the existence of a further constant of motion. Equilibria (15) were considered in a numerical model in [10].

With (c) a spread of betatron and revolution frequencies due to finite oscillation amplitudes is disregarded in our model. Thus dispersion is introduced in our system of particles only by the spread due to P_θ. It seems reasonable to assume that a spread of frequencies due to a finite range of betatron amplitudes has an effect upon Landau damping and stability which is comparable to the effect of an equally large spread produced by a finite range of P_θ. Moreover, there is some evidence to assume that both effects occur additively in the stability criteria [4].

3. Linear Perturbation Theory and Moment Equations

Small perturbations about the equilibrium (15)

$$(17) \quad f = f^0 + f^1$$

are treated with the linearized version of (1).

Using the independence of f^0 of t, θ, we may Fourier analyze f^1 with respect to these variables. The use of two further constants $H_{\perp r}^0$, $H_{\perp z}^0$ in f^0 suggests to Fourier analyze also with respect to the phase angles in $x - u_r$ and $z - u_z$. Here, however, we are only interested in modes with density variations in θ direction and beam displacement in r or z, corresponding to zero or first harmonics in the transverse phase planes. Higher transverse harmonics are neglected here, because they cause oscillations of the beam cross section which in turn have a minor effect on the collective field if γ is large.

With u_θ replaced by P_θ according to (6) we assume

$$(18) \quad f^1 = f^1(x,u_r,z,u_z,P_\theta)e^{i(l\theta-\omega t)}$$

and find from (1)

$$(19) \quad \begin{aligned} & il\frac{u_\theta}{r\gamma}f^1 - i\omega f^1 + \frac{u_r}{\gamma}\frac{\partial f^1}{\partial r} + \left(\frac{u_\theta^2}{r\gamma} + \frac{e}{m}(E_r^0 + \frac{u_\theta}{\gamma}B_z^0)\right)\frac{\partial f^1}{\partial u_r} + \frac{u_z}{\gamma}\frac{\partial f^1}{\partial u_z} \\ & + \frac{e}{m}(E_z^0 - \frac{u_\theta}{\gamma}B_r^0 + \frac{u_r}{\gamma}B_\theta^0)\frac{\partial f^1}{\partial u_z} = \\ & -\frac{e}{m}(E_r^1 + \frac{u_\theta}{\gamma}B_z^1 - \frac{u_z}{\gamma}B_\theta^1)\frac{\partial f^0}{\partial u_r} - e(E_\theta^1 - \frac{u_r}{\gamma}B_z^1 + \frac{u_z}{\gamma}B_r^1)r\frac{\partial f^0}{\partial P_\theta} \\ & -\frac{e}{m}(E_z^1 - \frac{u_\theta}{\gamma}B_r + \frac{u_r}{\gamma}B_\theta^1)\frac{\partial f^0}{\partial u_z} \end{aligned}$$

where we have used that

$$(20) \quad \begin{aligned} & \frac{\partial f^0}{\partial u_\theta} = m\, r\, \frac{\partial f^0}{\partial P_\theta}; \quad \frac{\partial f^1}{\partial u_\theta} = m\, r\, \frac{\partial f^1}{\partial P_\theta} \\ & \left.\frac{\partial f^1}{\partial r}\right|_{u_\theta} = \left.\frac{\partial f^1}{\partial r}\right|_{P_\theta} - \frac{\partial f^1}{\partial u_\theta}\frac{\partial u_\theta}{\partial r} \end{aligned}$$

and

$$(21) \quad \dot{u}_\theta \frac{\partial f^1}{\partial u_\theta} - \dot{r}\frac{\partial u_\theta}{\partial r}\frac{\partial f^1}{\partial u_\theta} = \dot{P}_\theta \frac{\partial u_\theta}{\partial P_\theta}\frac{\partial f^1}{\partial u_\theta} = 0$$

in first order, because P_θ is an equilibrium constant of motion.

From (19) we derive first order expressions for the momenta $\int f^1\, dr\, du_r\, dz\, du_z$, $\int x f^1\, dr\, du_r\, dz\, du_z$ and $\int z f^1\, dr\, du_r\, dz\, du_z$ which we need for calculating the beam displacement and current modulation in terms of f^1. Observing the subsequent relations following from (6), (10), (11)

$$(22) \quad \frac{\partial}{\partial u_{r,z,\theta}}\ \frac{1}{\gamma} = -\frac{u_{r,z,\theta}}{\gamma^3}$$

$$(23) \quad \frac{\partial x}{\partial P_\theta} = -\frac{dR}{dP_\theta}; \quad \frac{\partial u_\theta}{\partial P_\theta} = \frac{1}{mr}$$

(24) $$\frac{\partial}{\partial r}\frac{1}{\gamma} = -\frac{u_\theta}{\gamma^3}\frac{\partial u_\theta}{\partial r} = -\frac{e}{m}\frac{E_r^0}{\gamma^2}$$

(25) $$\frac{du_{o\theta}}{dP_\theta} = \frac{1}{mR} + \frac{e}{m}\frac{\gamma_o}{u_{o\theta}}\frac{dR}{dP_\theta}E_r^0$$

and in first order in x, z, u_r, u_z with $v_{o\theta} \equiv \frac{u_{o\theta}}{\gamma_o}$

$$u_\theta = u_{o\theta} + \frac{e}{m}\frac{\gamma_o}{u_{o\theta}}E_r^0(R,O)x$$

(27) $$\gamma = \gamma_o + \frac{e}{m}E_r^0 x$$

$$\frac{u_\theta}{\gamma} = v_{o\theta}\left(1 + \frac{e}{mv_{o\theta}^2\gamma_o^3}E_r^0 x\right)$$

We find from (19) after integrating over the transverse phase space with

$$d\sigma \equiv dr\, du_r\, dz\, du_z,$$

partial integrations and observing that f^0 is an even function of x, z, u_r, u_z

(28) $$-i\omega\int xf^1 d\sigma + il\int\frac{u_\theta}{r\gamma}xf^1 d\sigma - \int\frac{u_r}{\gamma}f^1 = -e\int rxE_\theta^1\frac{\partial f^0}{\partial P_\theta}d\sigma$$

(29) $$-i\omega\int u_r f^1 d\sigma + il\int\frac{u_\theta}{r\gamma}u_r f^1 d\sigma - \int\left(\frac{u_\theta^2}{r\gamma} - \frac{e}{m}\left(E_r^0 + \frac{u_\theta}{\gamma}B_z^0\right)\right)f^1 d\sigma = \frac{e}{m}\int\left(E_r^1 + v_{o\theta}B_z^1\right)f^0 d\sigma$$

(30) $$-i\omega\int f^1 d\sigma + il\int\frac{u_\theta}{r\gamma}f^1 d\sigma = -e\int rE_\theta^1\frac{\partial f^0}{\partial P_\theta}d\sigma$$

(31) $$-i\omega\int zf^1 d\sigma + il\int\frac{u_\theta}{r\gamma}zf^1 d\sigma - \int\frac{u_z}{\gamma}f^1 = 0$$

(32) $$-i\omega\int u_z f^1 d\sigma + il\int\frac{u_\theta}{r\gamma}u_z f^1 d\sigma - \frac{e}{m}\int\left(E_z - \frac{u_\theta}{\gamma}B_z^0\right)f^1 d\sigma = \frac{e}{m}\int\left(E_z^1 - v_{o\theta}B_z^1\right)f^0 d\sigma$$

With further use of (27) and elimination of $\int u_r f^1 d\sigma$, $\int u_z f^1 d\sigma$ we find the following expressions valid up to first order

(33) $$\int xf^1 d\sigma = \frac{i(\omega - l\Omega)R\frac{dR}{dP_\theta}eE_\theta^1 + \frac{e}{m\gamma_o}\left(E_r^1 + v_{o\theta}B_z^1\right)F^0}{\nu_r^2\Omega^2 - (\omega - l\Omega)^2}$$

(34) $$\int f^1 d\sigma = \frac{l\Omega}{\omega - l\Omega}\left(\frac{1}{R} - \frac{eE_r^0}{mv_{o\theta}^2\gamma_o^3}\right)\int x f^1 d\sigma - \frac{ie}{\omega - l\Omega}\frac{d}{dP_\theta}(E_\theta^1 R\, F^0)$$

(35) $$\int z f^1 d\sigma = \frac{\frac{e}{m\gamma_o}(E_z^1 - v_{o\theta}B_r^1)F^0}{\nu_z^2\Omega^2 - (\omega - l\Omega)^2}$$

(36) $$\Omega(P_\theta) \equiv \frac{v_{o\theta}}{R} \quad \text{(zero order gyro frequency)}$$

(37) $$F^0(P_\theta) \equiv \int f^0 d\sigma$$

(38) $$\nu_r^2(P_\theta) \equiv 1 - n\frac{B_{z\ ext}}{B_z + \frac{E_r^0}{v_{o\theta}}} + \frac{E_r^0 + R(E_r^{0\prime} + v_{o\theta}B_{z\ self}^{0\prime})}{E_r^0 + v_{o\theta}B_z^0} + \frac{(eE_r^0)^2R^2}{m^2v_{o\theta}^4\gamma_o^4}$$

(39) $$\nu_z^2(P_\theta) \equiv n\frac{B_{z\ ext}}{B_z^0 + \frac{E_r^0}{v_{o\theta}}} + \frac{R(E_z^{0\prime} - v_{o\theta}B_{r\ self}^{0\prime})}{E_r^0 + v_{o\theta}B_z^0}$$

(40) $$B_{r,z}^0 = B_{r,z\ ext} + B_{r,z\ self}^0$$

(41) $$n \equiv -\frac{R}{B_{z\ ext}}\frac{\partial B_{z\ ext}}{\partial r} \qquad \text{(magnetic field index)}$$

(All field terms are to be evaluated at $r = R$; $z = 0$ in (33)-(41)). The above defined ν_r^2, ν_z^2 are in agreement with the expression for the betatron frequencies derived in [11] analyzing the particle equations of motion in a stationary ring.

The local radial beam displacement as function of θ, t is defined by

(42) $$\langle r\rangle^1 = \frac{\int r f d\sigma dP_\theta}{\int f d\sigma P_\theta} - \frac{\int r f^0 d\sigma P_\theta}{\int f^0 d\sigma dP_\theta}$$

$$= \int\left(\int x f^1 d\sigma\right)dP_\theta + \int (R - \langle R\rangle)\left(\int f^1 d\sigma\right)dP_\theta$$

using the normalization

(43) $$\int f^0 d\sigma\, dP_\theta \equiv 1$$ and

(44) $$\langle R\rangle \equiv \int r f^0 d\sigma dP_\theta = \int R f^0 d\sigma dP_\theta .$$

The corresponding expression for the axial displacement is

$$(45)\quad \langle z\rangle^1 = \int\left(\int z f^1 d\sigma\right) dP_\theta$$

The next quantity necessary for calculating the collective field perturbations $\underline{E}^1$, $\underline{B}^1$ is the perturbation of the azimuthal component of the line currenct. Clearly, variations of the current density over the beam cross section are not relevant in a first order theory. Thus, from (4) with

$$du_\theta = \frac{dP_\theta}{mr}$$

$$(46)\quad \langle j_\theta\rangle^1 \equiv \frac{e}{m}\int \frac{u_\theta}{\gamma r} f^1 d\sigma dP_\theta$$

and with (11), (27)

$$(47)\quad \langle j_\theta\rangle^1 = \frac{e}{m}\int \Omega\left(f^1 d\sigma - \left(\frac{1}{R} - \frac{e}{m}\frac{E_r^{(0)}}{v_{o\theta}^2 \gamma_o^3}\right)\int x f^1 d\sigma\right) dP_\theta$$

The transverse components of the line current perturbation are written for completeness.

$$(48)\quad \langle j_r\rangle^1 \equiv \frac{e}{m}\int \frac{u_r}{\gamma r} f^1 d\sigma dP_\theta = \frac{e}{m}\int (\gamma_o R)^{-1}\left(\int u_r f^1 d\sigma\right) dP_\theta$$

$$(49)\quad \langle j_z\rangle^1 \equiv \frac{e}{m}\int \frac{u_z}{\gamma r} f^1 d\sigma dP_\theta = \frac{e}{m}\int (\gamma_o R)^{-1}\left(\int u_z f^1 d\sigma\right) dP_\theta$$

Thus we obtain from (33)-(35)

$$(50)\quad \langle r\rangle^1 = \int\left(1 - \frac{R - \langle R\rangle}{R}\frac{l\Omega}{\omega - l\Omega}\left(1 - \frac{eE_r^0 R}{v_{o\theta}^2 \gamma^3}\right)\right)\frac{i(\omega - l\Omega)R\frac{dR}{dP_\theta} eE_\theta^1 + \frac{e}{m\gamma_o}(E_r + v_{o\theta}B_z^1)}{\nu_r^2\Omega^2 - (\omega - l\Omega)^2} F^0 dP_\theta$$
$$- ie\int \frac{R - \langle R\rangle}{\omega - l\Omega}\frac{d}{dP_\theta}(E_\theta^1 R F^0) dP_\theta$$

$$(51)\quad \langle j_\theta\rangle^1 = -\omega\frac{e}{m}\int \frac{\Omega}{\omega - l\Omega}\frac{1}{R}\left(1 - \frac{eE_r^0 R}{v_{o\theta}^2 \gamma_o^3}\right)\frac{i(\omega - l\Omega)R\frac{dR}{dP_\theta} eE_\theta^1 + \frac{e}{m\gamma_o}(E_r^1 + v_{o\theta}B_z^1)}{\nu_r^2\Omega^2 - (\omega - l\Omega)^2} F^0 dP_\theta$$
$$- i\frac{e^2}{m}\int \frac{\Omega}{\omega - l\Omega}\frac{d}{dP_\theta}(E_\theta^1 R F^0) dP_\theta$$

$$(52)\quad \langle z\rangle^1 = \frac{e}{m}\int \frac{1}{\gamma_o}\frac{E_z^1 - v_{o\theta}B_r^1}{\nu_z^2\Omega^2 - (\omega - l\Omega)^2} F^0 dP_\theta$$

$$(53)\quad \langle j_r\rangle^1 = -i\frac{e}{m}\int\frac{\omega-l\Omega}{R}\,\frac{i(\omega-l\Omega)R\frac{dR}{dP_\theta}eE_\theta^1+\frac{e}{m\gamma_o}(E_r^1+v_{o\theta}B_z^1)}{\nu_r^2\Omega^2-(\omega-l\Omega)^2}F^0dP_\theta$$

$$+\frac{e^2}{m}\int E_\theta^1\frac{dR}{dP_\theta}F^0dP_\theta$$

$$(54)\quad \langle j_z\rangle^1 = -i\frac{e^2}{m^2}\int\frac{\omega-l\Omega}{R\gamma_o}\,\frac{E_z^1-v_{o\theta}B_r^1}{\nu_z^2\Omega^2-(\omega-l\Omega)^2}F^0dP_\theta$$

From these formulas we draw the following conclusions:

(a) The axial perturbations (52), (54) are in agreement with the corresponding expressions in [4], which were derived there for transverse oscillations of a straight beam.

(b) The radial perturbations (50), (53) differ from the straight beam expressions by additional terms involving the azimuthal electric field perturbation and the spread in equilibrium radii $\frac{R-\langle R\rangle}{R}$.

(c) The azimuthal current perturbation (51) has an additional term which is related to the radial beam displacement and introduces coupling between the radial and azimuthal coherent motions.

The relative importance of the different terms in (50)-(54) depends on the mode under consideration and will be discussed in the next chapter.

4. Dispersion Relations for Coherent Modes

The above established coupling between radial and azimuthal coherent motion depends quantitatively on the frequency ω of the mode. In the limit of vanishing collective field effects and no dispersion the zeros of the denominators in (50)-(54) permit us to classify the modes. For an observer gyrating with the particles a perturbing field $\simeq e^{i(l\theta-\omega t)}$ has a frequency $l\Omega-\omega$. This field may have a resonant action on the particles via

(a) their azimuthal motion ($\omega - l\Omega = 0$)

(b) their radial betatron oscillation ($(\omega - l\Omega)^2 - \nu_r^2\Omega^2 = 0$)

(c) their axial betatron oscillation ($(\omega - l\Omega)^2 - \nu_z^2\Omega^2 = 0$)

Next we assume finite, but not too strong self fields such that for either mode (a), (b) or (c) only those terms have to be maintained in (50)-(54), which are associated with the respective denominator.

We assume a linear dependence of the field perturbations at $r = \langle R\rangle$ on the quantities $\langle r\rangle^1$, $\langle j_\theta\rangle^1$, $\langle z\rangle^1$

(55) $E_r^1 + v_{o\theta} B_z^1 = F_{r,r}\langle r\rangle^1 + F_{r,\theta}\langle j\rangle^1$

(56) $E_\theta^1 = F_{\theta,r}\langle r\rangle^1 + F_{\theta,\theta}\langle j\rangle^1$

where the coefficients F depend on ω, l (in general) and follow from the Maxwell equations solved with the respective boundary conditions.

With the leading terms the dispersion relations are in the above defined cases:

(a) Azimuthal mode (negative mass mode)
The last integrals in (50), (51) are rewritten after partial integration.

$$\int \frac{R-\langle R\rangle}{\omega - l\Omega}\,\frac{d}{dP_\theta}(E_\theta^1 R F^0)\,dP_\theta = -\int\left(\frac{R-\langle R\rangle}{(\omega - l\Omega)^2}\,\frac{l\,d\Omega}{dP_\theta} + \frac{\frac{dR}{dP_\theta}}{\omega - l\Omega}\right)E_\theta^1 R F^0\,dP_\theta$$

(57)

$$\int \frac{\Omega}{\omega - l\Omega}\,\frac{d}{dP_\theta}(E_\theta^1 R F^0)\,dP_\theta = -\int\left(\frac{l\Omega\frac{d\Omega}{dP_\theta}}{(\omega - l\Omega)^2} + \frac{\frac{d\Omega}{dP_\theta}}{\omega - l\Omega}\right)E_\theta^1 R F^0\,dP_\theta$$

Evaluating E_θ^1 at $r = \langle R\rangle$ and discarding all contributions ecxept that with the denominator $(\omega - l\Omega)^2$ we find the dispersion relation for the negative mass instability.

$$(58)\quad 1 = iel\left[\frac{e}{m}F_{\theta,\theta}\int\frac{R\Omega\frac{d\Omega}{dP_\theta}}{(\omega-l\Omega)^2}F^0 dP_\theta + F_{\theta,r}\int\frac{R\,\frac{d\Omega}{dP_\theta}}{(\omega-l\Omega)^2}(R-\langle R\rangle)\cdot F^0 dP_\theta\right]$$

The second term expresses coupling to the radial motion through the variation in equilibrium radius and is in general small. The dominant first term agrees with the familiar negative mass dispersion relation [1,2] and we conclude that curvature has not an important effect on this mode as long as $(\omega-l\Omega) \ll l\Omega$ and $(\omega-l\Omega) \ll \nu_r\Omega$.

(b) Radial-azimuthal mode

Terms in (50)-(52) with the denominator $\nu_r^{\,2}\Omega^2-(\omega-l\Omega)^2$ contribute to this mode. Omitting the small term associated with $\frac{R-\langle R\rangle}{R}$ and neglecting

$$\frac{eE_r\,R}{v_{o\theta}^{\,2}\gamma_o^{\,3}}$$

compared with unity, we find

$$(59)\quad 1 = \frac{e}{m\gamma_o}\left\{i\frac{\omega-l\Omega}{\nu_r^{\,2}\Omega}F_{\theta,r}+F_{r,r}-\frac{e}{m}\omega\frac{\Omega}{\omega-l\Omega}\,\frac{1}{R}\left[i\frac{\omega-l\Omega}{\nu_r^2\Omega}F_{\theta,\theta}+F_{r,\theta}\right]\right\}\cdot D$$

$$D \equiv \int\frac{F^0}{\nu_r^{\,2}\Omega^2-(\omega-l\Omega)^2}dP_\theta$$

Here we have replaced for simplicity the Ω, R, ν_r by their average values, unless they occur in the denominator $\nu_r^{\,2}\Omega^2-(\omega-l\Omega)^2$, and have used the relation

$$(60)\quad \frac{dR}{dP_\theta} \simeq \frac{1}{m\gamma_o v_{o\theta}\nu_r^{\,2}}$$

which follows from (10).

(59) differs from the straight beam result [4] by two additional force terms in the brackets; the first term is relating the radial displacement with $E_\theta^{\,1}$ via the centrifugal force in the radial equation of motion; the third term includes the contribution of the azimuthal current modulation to the collective fields. Such a term appears because the azimuthal component of the particle motion obeys (see (27))

(61) $\dot{\theta} \simeq \Omega(1-\frac{x}{R})$

Hence it oscillates synchronous with the radial frequency and is driven therefore by the same oscillating collective field that drives the radial coherent motion. This term vanishes for $\omega \to 0$, because in this limit the local beam center moves perpendicular to the collective field which suppresses an azimuthal density modulation. In the limit of vanishing collective effects and dispersion the modes solving (59) are given by

(62) $\omega = (1 \pm \nu_r)\Omega$

The lower sign refers to the "slow wave", the upper sign to the "fast wave".

Inserting (62) into (59) gives

(63) $1 = \frac{e}{m\gamma_o}\{\pm i\frac{F_{\theta,r}}{\nu_r} + F_{r,r} - \frac{e}{m}\omega(\pm\frac{1}{\nu_r})\frac{1}{R}\left(\pm i\frac{F_{\theta,\theta}}{\nu_r} + F_{r,\theta}\right)\}\cdot D$

for either wave.

In the low frequency case, i.e. $1 \simeq \nu_r$, the slow wave leads to the dispersion relation

(64) $1 = \frac{e}{m\gamma_o}\{\frac{-i}{1}F_{\theta,r} + F_{r,r}\}D.$

We solve (64) for a ring with the realistic distribution

(65) $F^0(P_\theta) \simeq \left(1-(2\frac{P_\theta-<P_\theta>}{\Delta P_\theta})^2\right)$ for $|P_\theta-<P_\theta>|<\frac{\Delta P_\theta}{2}$ otherwise $F^0=0$

and find after evaluating D in essentially the same way as in [3] a criterium for Landau damping

(66) $\Delta S \gtrsim 3U$

where ΔS is the full spread of the quantity

(67) $S \equiv (1-\nu_r)\Omega$

due to the spread in P_θ and U is real and here defined by

(68) $e\,\{-\frac{i}{l}F_{\theta,r}+F_{r,r}\} = 2\,m\gamma_o\Omega\nu_r\big(U+(1+i)V\big).$

$V > 0$ is induced by finite resistivity of the walls due to image current damping [4]. U is interpreted as the shift of the coherent frequency due to collective effects in the limit of vanishing dispersion and $V \ll U$ as follows from (64), (68), i.e.

(69) $\omega = S + U$

Thus criterium (66) requires that the distribution in S is broad enough to cover the shifted frequency ω, otherwise there are no particles in phase with the coherent wave and there is no Landau damping.

With the following definition for the Landau damping coefficient

(70) $K = \frac{E}{\omega}\frac{dS}{dE}, \qquad \Delta S = K\,\frac{\Delta E}{E}\omega$

(66) is converted into a criterium involving the energy spread $\frac{\Delta E}{E}$ and K:

(71) $|K| \gtrsim \dfrac{3U}{\frac{\Delta E}{E}\,\omega} \simeq \dfrac{3\,U}{\frac{\Delta E}{E}(\langle S\rangle+U)}$

For most of the electron ring applications $|K|$ is in the range 0 ....2 and we conclude that Landau damping in the low frequency case is possible only if the average value of S obeys

(72) $\langle S\rangle \gg U,$

otherwise the mode is unstable ($V > 0$, finite resistivity) or purely oscillatory ($V = 0$). We observe that the same arguing holds if the considered mode is driven by resonant interaction with a ring of oppositely charged particles, which renders criterium (71) a necessary criterium for stabilizing the electron-ion instability *).

For a ring in free space with the mode $l = 1$ and $\omega \ll \Omega$ we apply the results for the self-fields of a stationary ring [12] on the coherently shifted ring and find in the unshifted coordinate system

(73)
$$e(E_r^1 + v_{o\theta} B_z^1) = m\gamma_o \Omega^2 \mu \left(-\frac{4R^2}{a(a+b)} \frac{1}{\gamma_o{}^2} + 2\ln\frac{16R}{a+b} \right) \langle r \rangle^1$$

$$e\, E_\theta^1 = -im\gamma_o \Omega^2 \mu \ln\frac{16R}{a+b} \langle r \rangle^1$$

a, b radial and axial semi axi

$$\mu = \frac{\nu}{\gamma} = \frac{Nr_e}{2\pi R\gamma}, \quad r_e \text{ classical electron radius}$$

Thus we find

(74)
$$U = \frac{\Omega}{2\nu_r} \mu \left(-\frac{4R^2}{a(a+b)} \frac{1}{\gamma^2} + \ln\frac{16R}{a+b} \right)$$

The logarithmic term in U is due to curvature and dominates for large γ over the straight beam term. As an example we take

$$\mu = 3\cdot 10^{-3}, \qquad \frac{R}{a} = \frac{R}{b} = 10, \qquad \gamma = 30$$

$$|K| \lesssim 2, \qquad \frac{\Delta E}{E} = 10^{-1}$$

and find that (71) can be satisfied only if

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*) A sufficient criterium must involve also dispersion of the ion species as observed already in [5].

(75) $(1-\nu_r) \gtrsim 10^{-1}$

whereas omission of the curvature term would allow for the much larger range $(1-\nu_r) \gtrsim 5 \cdot 10^{-3}$

(c) Axial mode

There is no coupling to the azimuthal motion and the dispersion relation shows the familiar result [4)]

$$(76) \quad 1 = \frac{e}{m\gamma_o} F_{z,z} \int \frac{F^0}{\nu_z{}^2 \Omega^2 - (\omega - l\Omega)^2} \, dP_\theta$$

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References

1 C.E. Nielsen, A.M. Sessler and K.R. Symon in Proc.Int. Conf.on High Energy Accelerators (CERN, Geneva, 1959), p.239

2 R.W. Landau and V.K. Neil, Phys.Fluids, 9, p.2412 (1966)

3 V.K. Neil and A.M. Sessler, Rev.Sci.Instr., 36, 429 (1965)

4 L.J. Laslett, V.K. Neil and A.M. Sessler, Rev.Sci.Instr., 36, 436 (1965)

5 D.G. Koshkarev and P.R. Zenkevich, Part.Accel., 3, 1 (1972)

6 U. Schumacher, C. Andelfinger and M. Ulrich, IEEE Trans. Nucl.Sci., NS-22, 989 (1975)

7 E.S. Weibel, Plasma Physics, 9, 665 (1967)

8 W.H. Kegel, Plasma Physics, 12, 105 (1969)

9 R.C. Davidson and J.D. Lawson, Part.Accel., 4, 1 (1972)

10 I. Hofmann, Proc.9th Int.Conf.on High Energy Accelerators (Stanford, 1974), p.245

11 M. Reiser, Part.Accel., 4, 239 (1973)

12 L.J. Laslett, Techn.Report ERAN 30 (1969), Lawrence Rad. Lab., Berkeley, California